Accuracy of a linear interpolation of the lunar distance

Jorrit Visser, CelNav.nl

2018-06-03

When comparing the measured lunar distance with the calculated lunar distance to determine the time, a nautical almanac or a table with precomputed lunar distances is used, which have data with one- or three-hour intervals. To calculate the lunar distance during these time intervals, a linear interpolation is done. When the moon passes a celestial body close by, the lunar distance does not change linearly with time. The question is how this affect the accuracy of the linear interpolation. In Figure 1 an example is shown where the lunar distance does not change linearly wih time.

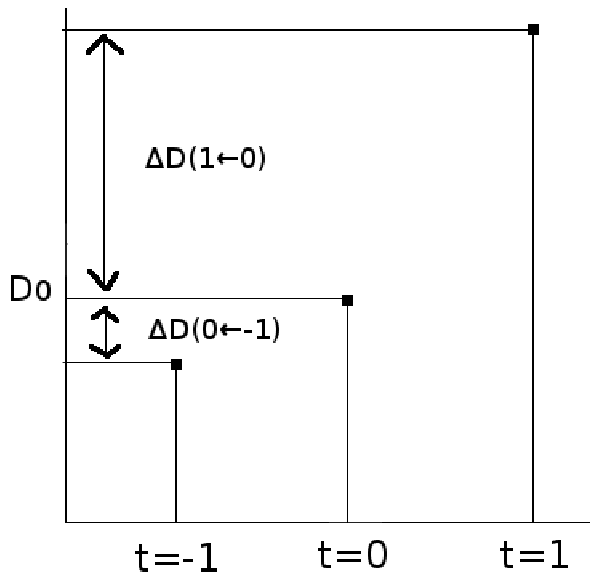

Figure 1: Lunar distance D at the previous ($t = -1$), current ($t = 0$), and next hour ($t = 1$). D_0 is the lunar distance at the current hour, $\Delta D(1 \leftarrow 0)$ is the difference in lunar distance between the next and current hour, and $\Delta D(0 \leftarrow -1)$ is difference in lunar distance between the current and previous hour.

If $D(t)$ is the lunar distance at time t, it follows from Figure 1 that:

$$D(0) = D_0 \tag{1}$$

$$D(-1) = D_0 - \Delta D(0 \leftarrow -1) \tag{2}$$

$$D(1) = D_0 + \Delta D(1 \leftarrow 0) \tag{3}$$

A linear interpolation of the lunar distance between the current and next hour can be expressed as follows:

$$D_{\text{lin}}(t) = D_0 + \Delta D(1 \leftarrow 0)t \tag{4}$$

For a quadratic interpolation we have:

$$D_{\text{quad}}(t) = D_0 + a_1 t + \frac{1}{2} a_2 t^2 \tag{5}$$

In this equation the parameters a_1 and a_2 are to be determined. If we fill in $t = 1$ and $t = -1$ and use (2) and (3) we find:

$$D_{\text{quad}}(1) = D_0 + a_1 + \frac{1}{2} a_2 = D_0 + \Delta D(1 \leftarrow 0) \tag{6}$$

$$D_{\text{quad}}(-1) = D_0 - a_1 + \frac{1}{2} a_2 = D_0 - \Delta D(0 \leftarrow -1) \tag{7}$$

It follows that:

$$a_1 = \frac{1}{2} \left[\Delta D(1 \leftarrow 0) + \Delta D(0 \leftarrow -1) \right] \tag{8}$$

$$a_2 = \Delta D(1 \leftarrow 0) - \Delta D(0 \leftarrow -1) \tag{9}$$

Thus a quadratic interpolation is expressed as follows:

$$D_{\text{quad}}(t) = D_0 + \frac{1}{2} \left[\Delta D(1 \leftarrow 0) + \Delta D(0 \leftarrow -1) \right] t + \frac{1}{2} \left[\Delta D(1 \leftarrow 0) - \Delta D(0 \leftarrow -1) \right] t^2 \tag{10}$$

Now we would like to determine the maximum difference between the quadratic and linear interpolation. The larger this difference is, the less accurate a linear interpolation of the lunar distance. From (4) and (10) it follows that:

$$D_{\text{quad}}(t) - D_{\text{lin}}(t) = \frac{1}{2} t \left(t - 1 \right) \left[\Delta D(1 \leftarrow 0) - \Delta D(0 \leftarrow -1) \right] \tag{11}$$

Setting the derivative with respect to t to zero:

$$\frac{\mathrm{d}}{\mathrm{dt}} \left[D_{\text{quad}}(t) - D_{\text{lin}}(t) \right] = 0 \Rightarrow t = \frac{1}{2} \tag{12}$$

This means that the maximum difference between the quadratic and linear interpolation occurs halfway between the current and next hour. The maximum difference ϵ_{max} is equal to:

$$\epsilon_{\text{max}} = \left| D_{\text{quad}}(1/2) - D_{\text{lin}}(1/2) \right| = \frac{1}{8} \left| \Delta D(1 \leftarrow 0) - \Delta D(0 \leftarrow -1) \right| \tag{13}$$

The difference between the quadratic and linear interpolation is a measure of the nonlinearity of the lunar distance over time. In Table 1 the lunar distance of Venus is given for May 17th, 2018, with one-hour intervals. On this day the moon passed close by Venus and the lunar distance of Venus changed nonlinearly over time. The maximum error that was made by a linear interpolation of the lunar distance was $0.5'$ of arc, as compared to a quadratic interpolation.

Table 1: On May 17th, 2018, the moon passed close by Venus. In this table the lunar distance of Venus is given at the whole hour. Also the difference between the lunar distance of the next and the current hour is given, as well as the maximum error of a linear interpolation of the lunar distance, as compared to a quadratic interpolation. Around 19 UTC, the moon was closest to Venus and the maximum error of a linear interpolation was about $0.5'$ of arc.

Hour (UTC)	Lunar distance	ΔD(next $\leftarrow$ current)	$\epsilon_{\max}$
4	9°39.2′	-29.0′	0.06′
5	9°10.2′	-28.4′	0.07′
6	8°41.7′	-27.8′	0.08′
7	8°14.0′	-27.0′	0.10′
8	7°47.0′	-26.1′	0.12′
9	7°20.9′	-25.0′	0.14′
10	6°56.0′	-23.7′	0.16′
11	6°32.3′	-22.1′	0.20′
12	6°10.2′	-20.2′	0.23′
13	5°50.0′	-18.0′	0.28′
14	5°32.0′	-15.4′	0.32′
15	5°16.5′	-12.5′	0.37′
16	5°04.1′	-9.1′	0.42′
17	4°55.0′	-5.4′	0.46′
18	4°49.5′	-1.6′	0.48′
19	4°48.0′	2.4′	0.49′
20	4°50.3′	6.2′	0.48′
21	4°56.6′	9.8′	0.45′
22	5°06.4′	13.1′	0.41′
23	5°19.5′	16.0′	0.36′